A Puzzle about Infinite Sums

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A famous mathematical theorem says that the sum of an infinite series of numbers can depend on the order in which those numbers occur. Suppose we interpret the numbers in such a series as representing instances of some physical quantity, such as the weights of a collection of items. The mathematics seems to lead to the result that the weight of a collection of items can depend on the order in which those items are weighed. But that is very hard to believe. A puzzle then arises: How do we interpret the metaphysical significance of this mathematical theorem? I first argue that prior solutions to the puzzle lead to implausible consequences. Then I develop my own solution, where the basic idea is that the weight of a collection of items is equal to the limit of the weights of its finite subcollections contained within ever-expanding regions of space. I show how my solution is intuitively plausible and philosophically motivated, how it reveals an underexplored line of metaphysical inquiry about quantities and locations, and how it elucidates some classic puzzles concerning supertasks, including the Ross-Littlewood Paradox and Thomson's Lamp.

§1 A Puzzle

The following principle initially seems unremarkable, yet turns out to be puzzling:

> **SUMMATION**
> For any collection of items, the weight of the collection equals the sum of the weights of the items within that collection.[1]

If the collection contains only finitely many items, then SUMMATION is indeed a bit boring. But if the collection is infinite, then a puzzle arises. The

[1] Are collections sets? Well, sets are abstract objects, and abstract objects don't weigh anything. It may be better to think of collections as fusions or pluralities.

puzzle doesn't cast doubt on the truth of SUMMATION. Instead, it raises questions about how to interpret the principle in the first place. As a prelude, consider the following thought-experiment (call it *Infinite Scale*) from Linnebo [2020: 1]:

> **Infinite Scale**
>
> Suppose you have a scale that is capable of weighing infinitely many items and an infinite amount of weight. Suppose also that you have an infinite number of iron balls and an infinite number of balloons. The first ball weighs 1 kg, the second ball weighs $\frac{1}{3}$ kg, the third ball weighs $\frac{1}{5}$ kg, and so forth. The first balloon lifts $\frac{1}{2}$ kg, the second balloon lifts $\frac{1}{4}$ kg, the third balloon lifts $\frac{1}{6}$ kg, and so forth. Now, suppose you first place the 1 kg ball on the scale, then attach the $-\frac{1}{2}$ kg balloon, then add the $\frac{1}{3}$ kg ball, then attach the $-\frac{1}{4}$ kg balloon, and so forth. This infinite sequence of actions results in an infinite progression of weights and counterweights added to the scale. What is the weight of the scale once every item has been added?

To figure out the answer to the question, we need to sum the weights of all the individual items. The scale starts with nothing on it, so we start at 0. Then we add 1 (for the first ball), then subtract $\frac{1}{2}$ (for the first balloon), then add $\frac{1}{3}$, then subtract $\frac{1}{4}$, and so on. This infinite sequence of actions is modeled by the following infinite series, which is sometimes called the 'alternating harmonic series':

The Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = ln(2) \approx 0.69.$$

The equation says that when we sum the sequence of numbers on the left, the result will gradually converge to $ln(2)$, which is approximately 0.69.

Therefore, it seems reasonable to conclude that the weight on the scale at the end of the procedure is ~.69kg. But the puzzle arises when we ask the following question: what if we were to rearrange the items in the series?

A surprising result from mathematics is that merely rearranging the order of the terms in a series can result in convergence to a different sum. In other words, the sum of an infinite series sometimes depends on the order of its terms. In fact, an astonishing result is that by rearranging terms, we can make a series sum to any arbitrary real number, or even tend towards positive or negative infinity. This is known as "Riemann's Rearrangement Theorem," after its discovery by the 19th-century mathematician Bernhard Riemann.[2] Here's a statement of the theorem:

> **Riemann's Rearrangement Theorem**
> If an infinite series is conditionally convergent, then its terms can be rearranged so that the new series converges to an arbitrary number, or diverges.

An infinite series *converges* iff the sum of its terms grows arbitrarily close to some finite number as the series progresses; otherwise, the infinite series *diverges*. An infinite series *conditionally converges* iff whether it converges depends on the order of its terms. The alternating harmonic series described above is an example of a conditionally convergent series: it sums to $ln(2)$, but if we rearrange its terms, we can generate a different sum. To do so, we take terms from the original series until the sum reaches the number we want to converge on, and then alternate between positive and negative terms from the original series so that the rearranged series converges to the desired limit. For the purposes of this paper, it isn't necessary to go deeper into the mathematical reasoning behind Riemann's Rearrangement Theorem, though interested readers may refer to the APPENDIX.

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[2] Riemann [1876].

The mathematical result is secure. The philosophical puzzle is how to interpret the metaphysical significance of that result. It's surprising that the sum of an infinite series depends on the order of its terms, but that may be regarded merely as a mathematical curiosity. What's much harder to believe is that the weight of a collection of items can depend on the order in which those items are weighed. Order of weighing seems a mere matter of convention, rather than an actual matter of metaphysics. And there's nothing special about weight: as I'll discuss later, analogous puzzles arise with any quantity that satisfies certain formal conditions. So, we have a metaphysical puzzle. Let's call it the *puzzle of conditional convergence*.

The aims of this paper are to (**1**) explain why existing solutions to the puzzle are unsatisfactory, (**2**) develop a new solution, (**3**) support my solution by investigating an underexplored collection of questions about quantities and locations, and (**4**) apply my results to some existing puzzles about supertasks. As we will see, the puzzle of conditional convergence isn't merely an isolated technical problem. Instead, the solution to the puzzle has much more general ramifications, yielding new lines of metaphysical inquiry and new tools for solving classic problems.

As for the puzzle itself, I'll argue that Riemann's Rearrangement Theorem indeed has interesting metaphysical implications. But it will take some work to uncover the exact nature of those implications. To solve the puzzle, we will need to consider the relationship between quantification over individuals and quantification over locations. Once we do so, the following solution will come to light: the weight of a collection of items (whether finite or infinite) is equal to the limit value of the weights of the finite subcollections contained within ever-expanding regions of space. The initial statement of this solution may feel a bit complex. But I'll argue over the course of the paper that the solution is intuitively plausible, philosophically motivated, and explanatorily fruitful.

You might be tempted to resist the coherence or relevance of infinitary scenarios. It's beyond the scope of this paper to fully address that perspective. But I'll make some brief remarks to address that perspective. It's an open possibility that the actual world is infinite. Given this, there may be actual facts of the matter about the relevant kinds of infinitary sums. But even if not, the puzzle of conditional convergence may be regarded as a way of motivating some more general observations about the relationships between quantities and locations. Those suspicious of the infinite may regard the thought-experiments as illustrative tools: as analogies, think about the role that scenarios like Hilbert's Hotel, philosophical zombies, and evil demon scenarios play in discussions of infinity, consciousness, and knowledge. Moreover, my ideas will be applicable to all quantities, even those that cannot generate puzzles of conditional convergence, as well as to classic puzzles about supertasks.

Here's the structure of the paper: **§2** argues against prior solutions to the puzzle, developed by Linnebo [2020] and Hoek [2021]; **§3** presents my view, which I call the 'expansionist analysis'; **§4** supports the expansionist analysis by exploring some general questions about quantities and locations; **§5** applies the observations to some puzzles concerning supertasks.

§2 The Order-Relative and Balance Analyses

Two solutions to the puzzle of conditional convergence already exist. The first, developed by Øystein Linnebo, is what I'll call the *order-relative analysis*: the weight of a collection depends on the order in which the individual items are weighed. The second solution, developed by Daniel Hoek, is what I'll call the *balance analysis*: the weight of a collection is zero whenever it contains both infinite positive weight and infinite negative weight. I'll argue that neither solution is satisfactory.

The Order-Relative Analysis

Since the puzzle was introduced by Linnebo [2020], it's fitting to start with his solution. Linnebo's view, in effect, is that we ought to take Riemann's Rearrangement Theorem at face value. According to his order-relative analysis, the weight of a collection of items depends on the order in which the individual items are weighed. The order-relative analysis predicts that in the scenario described above, the weight is approximately .69kg. But the order-relative analysis also holds that if the items were placed on the scale in a different order, then the result would be different.

The order-relative analysis is the most straightforward interpretation of the mathematical results. But the solution feels unsatisfying: it's very hard to believe that the order in which individual items are put on a scale can make a difference to the weight of the collection. In fact, we might wonder how weight works when our method of weighing doesn't involve any physical intervention on the items that are weighed. Suppose, for example, that we already know the weights of the individual items, the items are already lying on the ground, and we calculate the weight of the collection by inputting the weights of the items into a calculator. It's implausible that the weight of the collection depends on the order in which we enter numbers into the calculator. And if you and I were to enter the weights of the items in different orders, would that mean that the collection would then have multiple weight values? The reason these consequences feel absurd is because weight isn't a matter of mere bookkeeping; it's a real physical quantity. The solution I develop will preserve the order-invariance of weight (and other quantities).

To put pressure on the order-relative analysis, let's consider a variant on *Infinite Scale*. The variant invokes an application of Riemann's Rearrangement Theorem. Recall that the weights of the items in *Infinite Scale* were mathematically represented by the alternating harmonic series. Since that series is conditionally convergent, there's a rearrangement that diverges to ∞. Here's a procedure for achieving that result. We first separate the positive terms (which I'll label the p_i's) from the negative terms (which I'll label the n_i's). To construct the new series, we start with the first positive term p_1 (which is 1), followed by the first negative term n_1, followed by positive terms p_2, p_3, …, p_j until the sum is approximately 2, followed by the second negative term n_2, followed by positive terms p_{j+1}, p_{j+2}, …, p_k until the sum is approximately 3, and so forth. The result is a rearrangement of the alternating harmonic series with the following structure:

The Divergent Rearrangement

$$p_1 + (n_1 + p_2 + \ldots + p_j) + (n_2 + p_{j+1} + \ldots p_k) + \ldots = 1 + {\sim}1 + {\sim}1 + \ldots = \infty$$

Now we can construct a variant on *Infinite Scale*:

Clustered Items

Items: The same collection of items as in *Infinite Scale*.

Setup: The temporal order in which the items are placed corresponds to the alternating harmonic series (so the same as in *Infinite Scale*). But the spatial arrangement of the items corresponds to the divergent rearrangement (so an iron ball, followed by a balloon, followed by many iron balls, followed by a second balloon, followed by many, many iron balls, and so forth).

Clustered Items and *Infinite Scale* are indistinguishable with respect to both (1) the items that comprise the collection, and (2) the order in which the items are placed on the scale. They differ only with respect to (3) the spatial arrangement of those items. In *Clustered Items*, the iron balls are clustered together, and the balloons become sparse at an exponential rate; in *Infinite Scale*, every iron ball is adjacent to two balloons, and every balloon is adjacent to two iron balls. The order-relative analysis predicts that the weights of the collections are the same: namely, *ln*(2). But while that result is plausible for *Infinite Scale*, it's not at all obvious for *Clustered Items*. In fact, the most natural answer for *Clustered Items* is that the weight is ∞.[3] That's the

[3] Is ∞ a number? Well, this doesn't really matter for my arguments. The important point is that the weight in *Clustered Items* exceeds every finite weight value. Throughout the paper, I'll treat ∞ as a possible weight value, but I'll leave open how exactly

answer I'll eventually endorse. But for now, my point is modest: it's not obvious that we should assign *Clustered Items* and *Infinite Scale* the same weight value.

Here's another argument against the order-relative analysis, from Hoek [2021]. Consider a scenario where we start with all the items already placed on the infinite scale, and then remove them until the scale is empty:

Emptied Scale

Items: The same collection of items as in *Infinite Scale*.

Setup: We start at the end-state of *Infinite Scale*, where all items have already been placed and where (per the order-relative analysis) the resulting weight is .69kg. Then we remove the items as follows: first the 1kg ball *and* the $\frac{1}{3}$ kg ball, then the $-\frac{1}{2}$ kg balloon, then the $\frac{1}{5}$ kg *and* the $\frac{1}{7}$ kg ball, then the $-\frac{1}{4}$ kg balloon, and so forth until all items have been removed from the scale.

As Hoek notes, the natural generalization of the order-relative analysis will entail that the weight after all the items have been removed is a negative number. But that's absurd, since at the end of the procedure there's nothing on the scale. This is strong reason to reject the order-relative analysis.

The Balance Analysis

According to Hoek [2021], the weight of the collection in *Infinite Scale* is 0. He appeals to the following principle:

that is to be interpreted. If ∞ *is* treated as a number, then interesting questions arise about which trans-finite number system to deploy, but I won't discuss that here. See Easwaran *et al* [2021: §3] for discussion of related issues.

BALANCE

If equal weights and counterweights lie on a scale, then the scale is in the same state as when it holds no weights.

Call this view the *balance analysis*. The basic idea is that for any collection of items, we can partition it into two equivalence classes, one containing the positively weighted items and the other containing the negatively weighted items. If the sum of the positive weights is equal to the inverse of the sum of the negative weights, then the weight of the whole collection is 0. To motivate the balance analysis, Hoek reinterprets the infinite scale (where the counterweights are balloons) as an infinite balance (where the weights are on the left side and the counterweights are on the right side). Here's the relevant passage, from Hoek [2021: 2]:

Infinite Balance

Imagine an infinite stock of brass weights of 1kg, $\frac{1}{3}$kg, $\frac{1}{5}$kg, and so on; and an infinite stock of counterweights of $\frac{1}{2}$kg, $\frac{1}{4}$kg, $\frac{1}{6}$kg, and so on. At 1pm, we begin alternatively placing a weight on the left of our indestructible balance, and a counterweight on the right. We start with the biggest weights and work our way down...We speed up the steps as we go, so that at 2pm exactly, all steps have been performed. Which way will the balance lean after 2pm?

Since both the left side and the right side of the balance contain an infinite amount of weight, it seems plausible that the balance will be in equilibrium. But *Infinite Balance* seems to be merely a redescription of *Infinite Scale*. Therefore, if *Infinite Balance* is in equilibrium, then we ought to think that *Infinite Scale* outputs 0kg. Here's that argument in premise-conclusion form:

The Balance Argument

P1: *Infinite Balance* is in equilibrium.

P2: If *Infinite Balance* is in equilibrium, then the weight in *Infinite Scale* is 0.

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C: The weight in *Infinite Scale* is 0.

Hoek [2021] focuses mainly on defending P1. This is because the order-relative analysis denies P1. Or, more precisely, it's in the spirit of the order-relative analysis to accept that it's possible for *Infinite Balance* to lean in one direction or the other (rather than to be in equilibrium), depending on the order in which the individual items are weighed. To defend P1, Hoek argues against the following principle, which he takes to be the main motivation behind the order-relative analysis:

CONTINUITY

If a quantity converges to a limit x over time interval $[t_0, t_1)$, then the quantity attains value x at t_1.[4]

I won't argue against P1—I'll grant that *Infinite Balance* is in equilibrium. In fact, I agree that CONTINUITY doesn't hold in all cases, and in §5 I'll provide a diagnosis of when CONTINUITY works and when it doesn't. My goal, instead, will be to contest P2. On my view, *Infinite Balance* isn't merely an innocuous reinterpretation of *Infinite Scale*. The scenarios differ in ways that matter for how we assess the results for each case. As an initial problem for the balance analysis, consider the following principle:[5]

[4] Principles of continuity—and in particular, extensions from finite to infinitary cases—are often attributed to Leibniz. See Jorgensen [2009].

[5] Assume *a* and *b* don't overlap—I'll briefly discuss overlapping objects later, in §4.

FINITE ADDITIVITY
If *a* and *b* both have finite weight values, then the weight of *a* and *b* equals the weight of *a* plus the weight of *b*.

The balance analysis must deny FINITE ADDITIVITY. Let *a* be the collection of items in *Infinite Scale*, which the balance analysis says weighs 0, and let *b* be an additional iron ball weighing 1. Given FINITE ADDITIVITY, the weight of *a* and *b* should be 1. But the balance analysis instead predicts that the weight of *a* and *b* is 0. In fact, this holds no matter how much the additional item weighs, and no matter how many additional items we add. I don't take this consideration to be decisive; unexpected results often occur when dealing with the infinite. But I think the violation of FINITE ADDITIVITY is at least a strike against the balance analysis, especially since the mathematical analogue of FINITE ADDITIVITY holds even for conditionally convergent series.

Here's another challenge to the balance analysis:

Heavy Items

Items:	An infinite number of elephants, each of which weighs 5000kg. An infinite number of balloons, each of which lifts 0.01kg.
Setup:	An elephant is placed on the scale, then a balloon is attached to that elephant, then a second elephant is placed on the scale, then a second balloon is attached to that second elephant, and so on.

Suppose we apply the balance analysis to *Heavy Items*. First, we partition the weights (the elephants) from the counterweights (the balloons). Then, we place all the elephants on one side of an infinite balance. Since balloons have negative weights, we need to find a kind of item for the other side of the balance whose positive weights exactly balance the negative weights of balloons. Well, it's common knowledge that a standard helium balloon lifts approximately the weight of a slice of cheese. So imagine we replace each balloon with a slice of cheese, and then put all that cheese on the other side of the balance. Although one side contains elephants and the other side contains cheese, one might still reasonably conclude that the balance will be in equilibrium (since both sides contain an infinite amount of weight). But it's implausible that weight in *Heavy Items* is 0: instead, it's much more plausible that the weight is ∞. Therefore, even if we assume that *Infinite Balance* is in equilibrium, we ought not thereby infer that *Infinite Balance* outputs 0.

A proponent of the balance analysis might contend that our finite imaginative capacities are leading us astray. Just because any finite number of elephants and balloons has a positive weight doesn't mean that an infinite number of elephants and balloons likewise has a positive weight. As an analogy, consider the erroneous intuition that there are fewer prime numbers than integers (when, in fact, both sets have the same cardinality). However, this error-theoretic explanation is unlikely to be an apt diagnosis of the present case. Any finite number of elephants would outweigh the same number of slices of cheese—yet I granted above that an infinite number of elephants may very well weigh the same as an infinite number of slices of cheese. This is evidence that the intuition behind *Heavy Items* is sensitive to the asymmetries between finitary versus infinitary cases.

Besides, the argument can be strengthened. Here's a variant on *Heavy Items* that yields an especially forceful argument against the balance analysis:

Hungry Items

Items: An infinite number of elephants, each of which weighs 5000kg. An infinite number of balloons, each of which lifts 0.01kg.

Setup: An elephant is placed on the scale, then a balloon is fed to that elephant, then a second elephant is placed on the scale, then a second balloon is fed to that second elephant, and so on. Fortunately, the elephants have large gullets and flexible stomachs, so each

> balloon is swallowed whole by its elephant and remains inflated once inside its elephant's stomach.

The only difference between *Heavy Items* and *Hungry Items* is the relative locations of the elephants and balloons. In *Heavy Items*, the balloons are floating above the elephants; in *Hungry Items*, the balloons are inside the elephants. Unless we have independent reason for thinking that this change is relevant to the weights of the collections, we ought to treat *Heavy Items* and *Hungry Items* with parity. But surely the weight in *Hungry Items* is ∞. So we ought to think that the weight in *Heavy Items* is likewise ∞. This indicates that we ought to reject BALANCE (and, by consequence, the balance analysis).

We can now appreciate a more general problem for both the order-relative and balance analyses. Both solutions tacitly assume that we already know how to individuate the relevant items. However, when dealing with scenarios such as *Hungry Items*, it's not obvious how to do that. Should we count the balloons as separate from the elephants, or should each elephant (with a balloon inside) count as a single item? How we answer such questions will generate predictive differences for the order-relative and balance analyses. Yet there seem to be no non-arbitrary answers. And we cannot simply permit any method of individuation whatsoever, since doing so would lead to contradiction.

We have now evaluated two solutions to the puzzle. Both initially appeared promising, but both turned out to be vulnerable to compelling counterarguments. Let's now turn to the solution I favor.

§3 The Expansionist Analysis

Here's the basic idea behind my view. To solve the puzzle of conditional convergence, we need to know not only the weights of the individual items, but also their spatial arrangement. More precisely, we need to know whether the finite subcollections of items contained within ever-expanding regions of space always converge to the same weight value. If so, then that is the weight of the collection. Otherwise, the collection's weight is either infinite or undefined. I'll call this the *expansionist analysis*.[6]

The present section will focus mainly on explaining how the expansionist analysis works. But the full story—including the explanation for why weight and space are connected—will be developed also in §4, where I explore some more general metaphysical questions about quantities and locations. Some of the discussion in this section will be a bit technical—for those less interested in such points, it's possible to skim the technical details while still grasping the core ideas.

Definitions

Let a *ball* be the set of spatial points that lie within a given distance from some center.[7] To denote balls, I'll use the notation 'B(p, d)', where p is the ball's center and d is the ball's radius. For any collection A and ball B(p, d), we can identify the subcollection of A that lies inside B(p, d).[8] To denote this

[6] In the infinite ethics literature, there's a prominent view called 'expansionism' (for some discussions, see Vallentyne & Kagan [1997], Bostrom [2011], and Wilkinson [2020]). My view is similar in spirit, and my expansionist analysis is in some ways inspired by ideas from that literature. But it's important to appreciate that the target issues are distinct (even when we set aside the superficial difference concerning whether value or weight is the target quantity). The core difference is that expansionist theories in infinite ethics aim to yield *comparisons* between worlds with *infinite* total values, whereas my concern is with infinitary *collections* of items that sum to *finite* total values. Because of this, the main question of this paper is logically orthogonal to the main question examined in the infinite ethics literature. Still, my view is a natural complement to expansionist theories in infinite ethics. Moreover, some of my later arguments (especially in §4) may be taken to indirectly support those theories.

[7] It doesn't matter whether the ball is open (excluding its boundary points) or closed (containing its boundary points).

[8] What does it mean for an item a to be *inside* a region of space? Well, the answer doesn't matter. We can choose either (1) a is wholly inside the region, or (2) a is partially inside the region, or (3) a is mostly inside the region. Any of these choices will yield the same results.

subcollection, I'll use the notation '$\text{A}|\text{B}(p, d)$'. If no items of A lie inside $\text{B}(p, d)$, then $\text{A}|\text{B}(p, d) = \varnothing$; if every item of A lies inside $\text{B}(p, d)$), then $\text{A}|\text{B}(p, d) = \text{A}$.

Let ω be a function whose input is a finite collection of items and whose output is the weight of that collection. Hence, $\omega(\text{A}|\text{B}(p, d))$ is the weight of the finite subcollection of A that lies within the spatial ball with center p and radius d. For simplicity, I'll assume that any finite region of space contains only a finite number of weighted items (in §5, I'll discuss some cases where infinitely many items are in a finite region). Since we know how to calculate the weights of finite collections, and since any ball $\text{B}(p, d)$ is finite, there will always be a straightforward answer as to the value of $\omega(\text{A}|\text{B}(p, d))$.

Procedure

The expansionist analysis says that the weight of collection A is x iff the weights of A's subcollections contained inside ever-expanding balls always converge to x, no matter which spatial point those balls are centered on. More precisely, $\omega(\text{A}) = x$ iff for all spatial points p, $\omega(\text{A}|\text{B}(p, d))$ approaches the limit x as d tends to ∞.

Here's a procedure for determining whether that condition is satisfied. We start by picking an arbitrary spatial point p and an arbitrary distance d. These two values determine a ball $\text{B}(p, d)$—the ball with center p and radius d. This in turn determines a set of items $\text{A}|\text{B}(p, d)$—the subcollection of A that lies within $\text{B}(p, d)$. Then we ask: what happens to $\omega(\text{A}|\text{B}(p, d))$ as the ball grows larger?

To answer this, we define a sequence of balls that satisfies the following conditions: (1) every ball has the same center (namely, p), (2) every subsequent ball has a radius larger than the preceding ball, and (3) for every distance d_i, there's a ball in the sequence with radius d_n such that $d_n > d_i$. Put another way, the sequence of balls will be $(\text{B}(p, d_1), \text{B}(p, d_2), \text{B}(p, d_3), \ldots)$. The first term denotes the ball with center p and radius d_1, the second term denotes the ball with center p and radius d_2, and so forth. Hence, we have a sequence of ever-expanding balls, each centered on the point p.

We then use this sequence of balls to define a corresponding sequence of weights: in particular, the sequence (ω(A | B(p, d_1)), ω(A | B(p, d_2)), ω(A | B(p, d_3)), …). Here the first term denotes the weight of the subcollection of A that lies inside the ball with center p and radius d_1, the second term denotes the weight of the subcollection of A that lies inside the ball with center p and radius d_2, and so forth. Hence, we now have a sequence of the weights of the subcollections of A contained inside the sequence of ever-expanding balls (anchored on some center).

What we have defined so far is illustrated in the diagram below:

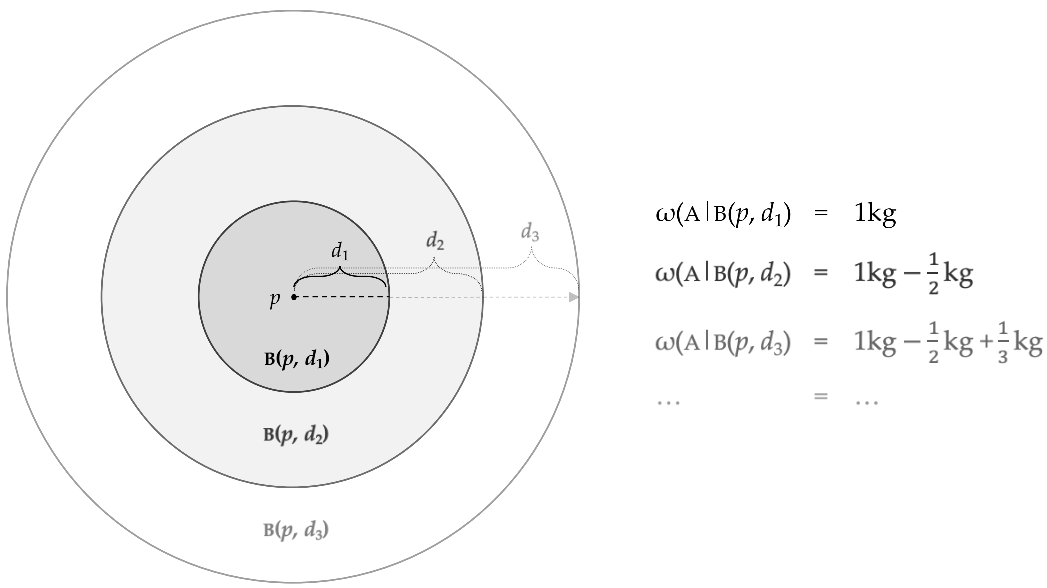

FIGURE 1: The weights of finite subcollections of A contained within ever-expanding spatial balls.

Let's call any sequence of weights derived through this kind of procedure an *ω-sequence*. The ω-sequence illustrated by the diagram above is (1kg, $\frac{1}{2}$kg, $\frac{5}{6}$kg, …).

For any collection A, we can generate a set of ω-sequences. For simplicity, let's suppose the distance intervals are always fixed (this won't make a difference in the end). Then the ω-sequences for any given collection will be individuated by which spatial point p marks the center of the balls.

In other words, for any collection A, there will be exactly one ω-sequence for every spatial point p. Now, for any ω-sequence, we can ask whether it approaches some limit x, meaning that the terms in the sequence get arbitrarily close to x as the sequence progresses. In other words, as the balls grow arbitrarily large, the weights of the subcollections of A within the balls become arbitrarily close to x.

Now I can state the core claim of the expansionist analysis. If every ω-sequence for A approaches the limit x, then the weight of A is x. If not, then the weight of A is either ±∞ or undefined. For the moment, I'll assume that every item in A is eventually captured by the ω-sequence (meaning that every item will eventually be contained within the ever-expanding sequence of balls). I'll later explain how to deal with cases where we drop that assumption.

The Expansionist Analysis

The expansionist analysis can be elegantly expressed with an equation (as reminders, A is a collection of items, x is a real number, and $\omega(\mathrm{A} \mid \mathrm{B}(p, d))$ is the weight of the subcollection of A that lies within the ball that has center p and radius d):

The Expansionist Analysis

$$\mathrm{A}\ \textit{weighs}\ x \equiv \forall p \lim_{d \to \infty} \omega(\mathrm{A} \mid \mathrm{B}(p, d)) = x$$

The analysis says that the weight of a collection A is x iff for every spatial point, if we consider an ever-expanding sequence of balls centered on that point, then the weights of the finite subcollections of A contained with those balls will approach x. That's equivalent to saying that all of A's ω-sequences approach x.

To make sure that the background mathematics is clear, it's worth making some remarks about the relationship between sequences and series. A sequence is an ordered list of terms; a series is the sum of the terms of an infinite sequence. What it means for a series to converge is for its sequence of partial sums to approach a limit. If, for example, the series is $a + b + c + \ldots$, then its sequence of partial sums is (a, $a + b$, $a + b + c$, …). If that sequence approaches a finite limit x, then the sequence (and the associated series) converges to x. Otherwise, the sequence diverges. Given this, there's a natural connection between the mathematical procedure for calculating a series and my metaphysical analysis of weight: the ω-sequences defined above are the relevant sequences of partial sums.

In fact, each of the solutions to the puzzle of conditional convergence offers a different answer as to which sequences of partial sums are relevant for determining the weight of a collection. The order-relative analysis says that the relevant sequence of partial sums corresponds to the order in which the items are weighed. The balance analysis identifies two relevant sequences of partial sums, one comprised of all the positively weighted items and the other comprised of all the negatively weighted items. The expansionist analysis takes the relevant sequences of partial sums to correspond to the weights of the finite subcollections contained inside ever-expanding balls. The fact that these different answers are available illustrates how the mathematics underdetermines the metaphysics. Finding a solution to the puzzle of conditional convergence is a matter of identifying the right metaphysical interpretation of the mathematics.

I've now explained how the expansionist analysis works for cases involving conditional convergence. But what about cases involving divergence? If some of A's ω-sequences approach limit x while others approach limit y, then the weight of A is undefined. If each of A's ω-sequences tends to ∞ (or $-\infty$). then the weight of A is ∞ (or $-\infty$). Furthermore, while the expansionist analysis is motivated by the puzzle of conditional convergence (which involves infinitary collections), it generalizes to finite cases as well. If A is finite, then every item in A will eventually be contained within any sequence of ever-expanding balls, so every ω-sequence will eventually converge to exactly the weight of A.

Some might wonder why the expansionist analysis appeals to *uniform* expansions from a center. In principle, we could consider a more permissive restriction on expansions, such as a convexity or connectedness restriction. But it's plausible that uniformity is the relevant restriction, at least in the analysis of weight. If we were to instead adopt a more permissive constraint on expansions, then we would be faced with implausible predictions. Consider, for example, *Hungry Items*. Since there are gruesome sequences of spatial expansions where the associated partial sums in this scenario don't diverge to ∞, adopting a more permissive restriction on expansions would predict that the weight in *Hungry Items* is undefined rather than ∞. This strikes me as a good reason for rejecting a more permissive restriction on expansions for weight. Now, there's an interesting question of whether there are other quantities for which non-uniform expansions are permissible. I suspect that when dealing with natural quantities, we will need to always restrict the analysis to natural expansions. But I won't explore this question further: if there *are* other quantities for which it's appropriate to appeal to more permissive restrictions on expansions, then it's easy enough to modify the expansionist analysis accordingly.

Isolated Items

The analysis above assumes that all the items in the collection are located within the same space. But what happens when we have isolated spaces? Let's say two items are *isolated* from each other if their distance is undefined. Imagine, for example, that a multiverse hypothesis is true, where there are infinitely many spatiotemporally isolated universes (all of which are actual). And consider a finite collection of isolated items:

Finite Isolation

Items: Two iron balls, *a* and *b*, where *a* weighs 1kg and is located in universe A, *b* weighs 2kg and is in universe B.

What's the weight of the collection comprised of *a* and *b*? Well, it's clearly 3. But no ever-expanding sequence of balls starting in *a*'s universe will ever reach any point in *b*'s universe, since the universes are spatially isolated. In fact, every ω-sequence starting in A will converge to 1, and every ω-sequence starting in B will converge to 2.

Fortunately, it's straightforward how to deal with this sort of case. To determine the weight of a collection that contains some pairs of isolated items, we partition the collection into a set of equivalence classes, where each equivalence class consists of all and only the items that aren't isolated from each other. Put another way, we group together items that inhabit the same space. Then we apply the expansionist analysis to each equivalence class, yielding a set of weight values. After that, we simply add up the weight values associated with each equivalence class to get a total weight value. In *Finite Isolation*, we apply the expansionist analysis twice—once to the subcollection in universe A, yielding a weight of 1, and again to the subcollection in universe B, yielding a weight of 2—and add up those weights, yielding a total weight of 3.

In the next section, after we examine more general connections between quantities and locations, we will have a deeper understanding of why the procedure above is justified. But here's a preview. There are two distinct ways of summing quantities: (1) over a collection of individuals, and (2) over a collection of locations. In some cases, such as *Infinite Scale*, summation over individuals doesn't yield a definite verdict, since different orderings of the terms yield different sums. Yet we can still appeal to summation over locations (using the expansionist analysis) to find an answer. Conversely, in other cases, such as *Finite Isolation*, summation over locations doesn't yield a definite verdict, since some of the items are isolated from each other. Yet we can still appeal to summation over individuals (using FINITE ADDITIVITY) to find an answer. In fact, even when we have an *infinite* collection of isolated items, we can still appeal to summation over the weight values of the individual items to find an answer, so long as the weight values either generate an absolutely convergent series or diverge to ±∞.

But what if the weight values of a collection of isolated items generate a conditionally convergent series?

Infinite Isolation

Items: The exact same collection of items as in *Infinite Scale*, with every item located in a spatiotemporally isolated universe.

My view is that the weight in *Infinite Isolation* is undefined. At least, I see no non-arbitrary way of summing the weights of the items in this scenario. This might elicit the worry that 'undefined' is a non-answer. Won't there be some fact of the matter as to the reading of the scale? Speaking for myself, I think it's better to treat the infinite scale scenarios as illustrative thought-experiments (like Hilbert's Hotel) rather than as genuine metaphysical possibilities. Moreover, just about everyone will accept that there are undefined weight values in at least some scenarios. Setting aside isolated items, we can construct scenarios involving bounded divergence, such as the sequence 1 − 2 + 3 − 4 + …. Those who reject 'undefined' as a possible answer must figure out what to say about these sorts of scenarios. On the other hand, if we accept 'undefined' as the right answer in these cases, it's reasonable to think that it likewise applies in *Infinite Isolation*.

Predictions

Let's return to the big picture. I'll now explain how the expansionist analysis does better than the order-relative and balance analyses with respect to predictions about cases. Here are the scenarios that have occurred throughout the paper (in the order in which they appeared), alongside the predictions of the expansionist analysis:

Scenario	**Prediction (in *kg*)**
Infinite Scale	.69
Clustered Items	∞
Emptied Scale	0
Infinite Balance[9]	Equilibrium
Heavy Items	∞
Hungry Items	∞
Finite Isolation	3
Infinite Isolation	Undefined

I've already discussed the isolation cases. I'll briefly walk through the other predictions, starting with *Infinite Scale*.

Strictly speaking, *Infinite Scale* is under-described, since the scenario didn't specify the spatial arrangement of the items. But suppose the spatial arrangement of items corresponds to the temporal order in which those items are placed. Imagine, for example, that the first item is placed all the way on the left, the second item to the right of the first, the third item to the right of the second, and so forth. Then the expansionist analysis plausibly predicts (alongside the order-relative analysis) that the weight of *Infinite Scale* is .69. Next, consider *Infinite Balance*. It's uncontroversial that the weight on each side of the balance is ∞. If we then grant that a balance is in equilibrium just in case both sides carry the same amount of weight,[10] then we reach the result that *Infinite Balance* is in equilibrium.

The counterexamples to the order-relative analysis were *Clustered Items*, in which we arranged the items from *Infinite Scale* so that the iron

[9] Here I interpret *Infinite Balance* as involving two collections of weights, one on either side of the balance.

[10] Actually, I don't find this principle obvious. In the infinite ethics literature, many deny the analogous principle concerning value (so that even if worlds A and B each have value ∞, it may nevertheless be that A is better than B). It seems to me a live possibility that analogous considerations apply to other quantities as well.

balls were clustered together, and *Emptied Scale*, in which we removed all the items from *Infinite Scale* in a different order than they were placed. The order-relative analysis predicted that the weight of *Clustered Items* was .69 (instead of ∞) and that the weight on *Emptied Scale* would be negative (instead of 0). By contrast, the expansionist analysis plausibly predicts that the weight in *Clustered Items* is ∞ (since sequences of ever-expanding balls will generate sequences of ever greater weight) and that the weight in *Emptied Scale* is 0 (since there are no items on the scale when the scale is empty). Moreover, whereas the order-relative analysis takes weight to depend on what seems to be a mere matter of convention, we will soon see that the expansionist analysis appeals to independently motivated connections between quantities and locations.

The counterexamples to the balance analysis were *Heavy Items*, where infinitely many elephants had balloons attached to their backs, and *Hungry Items*, where infinitely many elephants had balloons inside their stomachs. The balance analysis predicted that the weight in both *Heavy Items* and *Hungry Items* is 0, since both scenarios involved both infinite positive weight and infinite negative weight. By contrast, the expansionist analysis plausibly predicts that the weight in both cases is ∞ (since in both cases, all sequences of ever-expanding balls generate sequences of ever greater weight). Moreover, whereas the balance analysis had to give up FINITE ADDITIVITY, the expansionist analysis can retain that principle..

Although I've argued that neither the order-relative nor the balance analyses are correct, I think that both still get something fundamentally right. The order-relative analysis is fundamentally correct that Riemann's Rearrangement Theorem has metaphysical implications: by simply rearranging the items within a collection, we can change the weight of that collection. However, the order-relative analysis misidentifies the relevant parameter of rearrangement: it's spatial distribution, rather than temporal order, that matters. The balance analysis is fundamentally correct in rejecting CONTINUITY: just because the weights in a sequence of finite subcollections approaches the value x before time t doesn't mean that the resulting infinite collection at time t weighs x. However, the balance analysis is too quick to dismiss the relevance of convergence to limits: it's convergence over regions of space, rather than intervals of time, that matters for weight. The expansionist analysis incorporates these lessons.

Some might still feel uneasy about taking the weight of a collection of items to depend on the spatial arrangement of those items. I agree that this consequence feels surprising. But everyone must accept some surprising consequences about infinitary scenarios, and I believe this is a consequence that we can get used to, learn to live with, and perhaps even come to love. Besides, it's a consequence that just about everyone will want to accept in at least some cases. Imagine a scenario where 1kg iron balls occupy odd-numbered slots and −1kg balloons occupy even-numbered slots, and contrast that with a scenario where 1kg iron balls occupy composite-numbered slots and −1kg balloons occupy prime-numbered slots. Since the occurrence of prime numbers grows increasingly infrequent as we move along the natural number line,[11] the positive weight is much more densely distributed in the first scenario than in the second. Yet the two scenarios may be thought of as mere spatial rearrangements, since every item in one scenario can be mapped to a corresponding item in the other scenario. If one denies any connection between weight and space, then one will be forced to say that these scenarios involve equal weight values. That strikes me as much more costly than the idea that weight can depend on spatial arrangement.

Some might feel the allure of the expansionist analysis yet remain puzzled about why it might be true. The next section will develop a deeper justification for the analysis.

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[11] This follows from the prime number theorem, proved by Hadamard and de la Vallée Poussin in 1896. Interestingly, this theorem was also based on work by Riemann (in particular, the Riemann zeta function). See Weisstein [2022].

§4 Quantities and Locations

In the expansionist analysis, space plays the role of the *locative category*: to determine the weight of a collection, we consider ever-expanding regions of space (rather than time, spacetime, or something else). Yet nothing in the formalism necessitates an appeal to *space*: in principle, we could have instead appealed to ever-expanding temporal intervals, ever-expanding light cones, or ever-expanding regions of some other kind. In fact, anything with metric structure—the kind of structure associated with distances between elements—could be used for the locative category. So what justifies the connection between weight and space?[12]

This section addresses that question. I'll also aim to show how finding the answer to that question points to some novel questions about quantities and locations. At some points, I'll offer more questions than answers. But I think that will simply serve to illustrate that much of the metaphysical terrain is underexplored.

Category Mistakes

Consider an asymmetry: it makes sense to ask how much weight is in a given region of space, but it doesn't make sense to ask how much weight is in a given region of time. Contrast 1*a* with 1*b*–1*d*:[13]

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[12] The ensuing discussion will assume a classical picture of space and time. One reason is merely to simplify the discussion. But another reason is that many of our concepts—including WEIGHT—are arguably classical concepts. That is, regardless of the actual metaphysics of space and time, WEIGHT bears different conceptual relations to SPACE than it does to TIME. Now, this raises some interesting questions about conceptual engineering: how should we adjust our concept of weight when we move to a relativistic framework? The natural options are to (1) appeal to frame-variant regions of space, or (2) appeal to frame-invariant regions of spacetime. I won't attempt to settle which of these options is best.

[13] These sentences are all formulated as questions, but other syntactic constructions (such as declarative sentences) would work just as well.

(1*a*) How much weight is in this room?
\# (1*b*) How much weight is in this hour?
\# (1*c*) How much weight is in the red region of color space?
\# (1*d*) How much weight is in the interval (0, 1)?

Whereas 1*a* is a sensible question, 1*b*–1*d* are category mistakes:[14] weight can be instantiated at regions of space, but it can't be instantiated at regions of other sorts of spaces. This asymmetry with respect to locations is analogous to a more familiar asymmetry with respect to individuals. Only certain kinds of entities—namely, material objects—have weight values. Other kinds of entities—social entities, abstract objects, feelings, etc.—aren't the sorts of things for which it makes sense to ascribe weight values. Consider the asymmetry between 2*a* and 2*b*–2*d*:

(2*a*) How much does Riemann weigh?
\# (2*b*) How much does the United Nations weigh?
\# (2*c*) How much does the number 3 weigh?
\# (2*d*) How much does love weigh?

Just as weight can be instantiated only *by* material objects, weight can be instantiated only *at* spatial locations. Even though the formalism for the expansionist analysis leaves open which locative category has the relevant metric structure, the interpretation of the formalism makes sense only if we take the locative category to be space. When examining a quantity, we can ask both a *what* question (what can instantiate the quantity?) and a *where* question (where can the quantity be instantiated?). The answer to the latter

[14] 1*c* and 1*d* are especially odd because they commit an extra category mistake: only concrete objects can have weights, but concrete objects don't occupy regions of color space (rather, colors do) or regions of the real line $\mathbb{R}$ (rather, numbers do).

question tells us which kinds of locations are relevant to the expansionist analysis for that quantity.[15]

Not every quantity has space as its locative category. Contrast weight with pain. While it doesn't makes sense to ask how much weight occurred over an interval of time, it *does* make sense to ask how much pain occurred over an interval of time. If one feels pain for a longer duration, then more pain is instantiated. This indicates that time is a locative category for pain, even though it isn't for weight. Or contrast weight and number-of-prime-integers. While it doesn't make sense to ask how many primes there are in a given region of physical space, it *does* make sense to ask how many primes there are in a given interval of the number line. This indicates that while space is a locative category for weight, it isn't for number-of-primes.

Now, in one sense, weight values *are* indexed to times. If we ask how much weight is in a given region R (or instantiated by a given collection A), then we must specify the time at which we're evaluating the weight of R (or A). Otherwise, there won't be a determinate answer to the question, since the weight of a given region or collection may vary across different times. However, the way in which weight is indexed to time is different from the way in which weight is indexed to space (and to material objects). As we saw above, it doesn't make sense to sum weight over intervals of time, whereas it does make sense to sum weight over regions of space or collections of individuals. The specification of a time fixes the context of evaluation, rather than the domain of summation. Weight values are specified at particular times, but they aren't summed over temporal intervals.

It may turn out that some quantities are *locationless*, meaning they lack a locative category. If a quantity Q is locationless, then there are no answers to questions of the form 'Where is Q instantiated?' and no true

[15] This point is especially relevant to expansionist theories in the infinite ethics literature. A common criticism of these theories is that it's not clear why spacetime is ethically significant. My arguments here point to a response: spacetime is the locative category for value.

sentences of the form 'Q is instantiated at R'. As a potential example, consider wealth. It's not obvious that it makes sense to ask how much wealth is instantiated within a given region of space or time (or any other locative category).[16] Similarly, it might turn out that some quantities are *objectless*, meaning they lack a category of individuals. As a potential example, consider number-of-prime-integers. While we can ask how many prime integers there are within a given region of the real line, it's not obvious that number of prime integers is instantiable by any individual entity (unless we interpret regions of the real line as themselves individuals).

If there are locationless quantities, then what happens when they generate puzzles of conditional convergence? Well, the expansionist analysis identifies restrictions on which sequences of partial sums are relevant for summation: in particular, the relevant partial sums are the values of the finite subcollections inside ever-expanding balls of the relevant locative category. If a quantity is locationless, then there are no such restrictions. So it's natural to think that for these quantities, *all* sequences of partial sums are relevant. By consequence, when we construct a conditional convergence scenario for a locationless quantity, the value of the collection will be undefined.

There's still a big question that remains unanswered: What exactly *is* a location? This is a hard question, and I don't have a settled answer. But I don't think we need one for present purposes. The connections between quantities and locations I've identified are compatible with a range of views about the nature of locations. In particular, my arguments are intended to leave open whether individuals and locations are mutually exclusive

[16] If your linguistic intuitions differ, make sure that you aren't interpreting such expressions as elliptically asking how much wealth is instantiated by the collection of people within a given region of space or at a particular time.

categories, whether locations are ultimately absolute or relational, and how to best develop a formal theory of locations.[17]

The methodology I've applied to weight can be generalized. If R is a region that belongs to the category of locations for Q, then we should be able to sensibly ask 'How much of quantity Q is instantiated within region R?' If A is a collection of individuals that all belong to the category of individuals for Q, then we should be able to sensibly ask 'What is the value of collection A with respect to quantity Q?' The answers to such questions provide evidence as to the target quantity's category of locations and category of individuals.

Metaphysical Principles

I'll now turn to some metaphysical principles connecting quantities, locations, and individuals. My aim is partly to illustrate some of the questions that arise when we pursue this line of metaphysical inquiry. But these principles will also bear on the puzzle of conditional convergence: the first principle will mitigate one of the dialectical burdens of the expansionist analysis, the second principle illustrates how summation over locations behaves in systematic ways, and the third principle is a generalization of the initial summation principle introduced at the very beginning of this paper.

Suppose A is a collection of items and R is a region of space that contains all (and only) those items. Recall that ω is a function that takes as input a collection of items and outputs the weight of that collection. Let's generalize ω so that it can also take as input a region of space (where the output would then be the weight contained within that region of space). Here's a plausible principle about how the weights of collections of individuals relate to the weights contained within locations:

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[17] For some work on the nature of locations, see Casati & Varzi [1998], Hawthorne & Sider [2002], Parsons [2007], and Kleinschmidt (*ed*) [2014].

QUANTIFICATION EQUALITY

If A is the collection of weighted items in region R, then $\omega(\text{A}) = \omega(\text{R})$.[18]

Our focus throughout the paper has been on questions about the weights of infinitary collections of items. But we could have instead focused on questions about the weights contained within infinitary regions of space. Let A be one of the infinitary collections of items we have considered (such as the collection in *Infinite Scale*) and R be the region of space that contains all and only those items. QUANTIFICATION EQUALITY entails that the answers to the following questions should be the same:

Q1: What is the weight of A?
Q2: What is the weight within R?

The equivalence of these questions matters for the puzzle of conditional convergence. According to the expansionist analysis, we can answer Q1 by finding the limit of the weights of the finite subcollections of A contained within ever-expanding regions of space. It's natural to then ask why regions of space are relevant for calculating the weight of a collection of individuals. But that question feels less compelling when we shift the focus from Q1 to Q2. If we ask how much weight is contained within some region R, it seems obvious that the answer can be found by summing the weights contained within the subregions of R. But QUANTIFICATION EQUALITY entails that Q1 and Q2 will have the same answer. Hence, if it's permissible to appeal to space to answer Q2, it should also be permissible to appeal to space to answer Q1.

[18] According to *supersubtantivalism*, material objects are identical to regions of spacetime. If supersubstantivalism is true, then QUANTIFICATION EQUALITY trivially follows. But even if supersubstantivalism is false, QUANTIFICATION EQUALITY remains plausible.

QUANTIFICATION EQUALITY also enables the expansionist analysis to avoid a problem that beset the order-relative and balance analyses. Recall that both analyses were subject to a problem about how to individuate the items within the collection. The problem was illustrated via *Hungry Items*, when we asked whether the elephants and the balloons counted as separate items or whether each elephant and the balloon inside of it counted as a single item. On the expansionist analysis, however, it doesn't matter how we answer such questions. The reason is due to QUANTIFICATION EQUALITY. For any region of space, there's some determinate answer as to the amount of weight contained within that region, no matter how we individuate the collection of items contained within that region.

The second principle concerns summation over the weights contained within regions. Suppose we already know the weights contained within two regions of space, R_1 and R_2. What, then, is the weight of the union of R_1 and R_2? The following answer is plausible:

SUMMATION OVER REGIONS
$\omega(R_1 \cup R_2) = \omega(R_1) + \omega(R_1) - \omega(R_1 \cap R_2)$.

This principle yields the right verdicts across different cases. There are three possibilities for how R_1 and R_2 may be related: (1) R_1 and R_2 are identical, (2) R_1 and R_2 are disjoint, and (3) R_1 and R_2 overlap (where the overlap is partial, and where this includes cases where one region wholly contains the other). Here's what the principle says for each case:

(1) **Identity:** The weight in $R_1 \cup R_2$ is the weight contained within either region. That is: $\omega(R_1 \cup R_2) = \omega(R_1) = \omega(R_2)$.

(2) **Disjointness:** The weight in $R_1 \cup R_2$ is the weight of R_1 plus the weight of R_1. That is: $R_1 \cap R_2 = \varnothing$, so $\omega(R_1 \cap R_2) = 0$, so $\omega(R_1 \cup R_2) = \omega(R_1) + \omega(R_1)$.

(3) **Overlap:** The weight in $R_1 \cup R_2$ is the disjoint part of R_1, plus the disjoint part of R_2, plus the intersection of R_1 and R_2. That is: $\omega(R_1 \cup R_2) = \omega(R_1 \setminus R_2) + \omega(R_2 \setminus R_1) + \omega(R_1 \cap R_2)$.

It's worth comparing SUMMATION OVER REGIONS to the corresponding principle concerning summation over individuals. We've already encountered a version of the latter principle: it was introduced at the beginning of the paper, under the simple label 'SUMMATION', and it stated that for any collection of items, the weight of the collection equals the sum of the weights of the items within that collection. That principle is plausible if we assume that none of the items in the collection overlap with each other. But we can also generalize that principle so that it applies even when the items overlap. Consider, for example, a statue and the clay that constitutes it, which are distinct but overlapping objects. If we are calculating the weights of material objects, then (in most contexts)[19] we wouldn't want to double-count the weight of the statue and the weight of the clay. Put another way, the weights of overlapping material objects are quantitatively redundant. To capture this precisely, we can construct a principle for summation over individuals that's structurally analogous to the principle for summation over regions. Whereas the second principle ranged over two regions R_1 and R_2, the third principle ranges over two individuals *a* and *b*:

SUMMATION OVER INDIVIDUALS
$\omega(a, b) = \omega(a) + \omega(b) - \omega(a \cap b)$.

An interesting question is how the intersection relation in SUMMATION OVER INDIVIDUALS relates to the intersection relation in SUMMATION OVER REGIONS.

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[19] There may be some unusual contexts where we would want to count the weights of the statue and the clay separately: perhaps, for example, in certain metaphysics seminars. But in these contexts, we would be deploying a non-standard method for summing weights.

One option is to hold that they are the same: for two individuals to intersect just is for them to intersect in their spatial locations. Another option is to hold that they differ: mereological intersection is distinct from spatial intersection. I won't take a stance on this issue. But it's worth noting that those who favor the second option face an explanatory challenge. Given QUANTIFICATION EQUALITY, SUMMATION OVER REGIONS, and SUMMATION OVER INDIVIDUALS, two individuals overlap just in case their spatial regions overlap. Those who posit two overlap relations must explain why these relations systematically coincide.

Generalizing the Puzzle

To generate a puzzle of conditional convergence, a quantity must be (1) *summative*, meaning that the quantity value of a collection is the sum of the quantity values of the individuals within that collection, (2) *convergeable*, meaning that the quantity values can be arbitrarily close to zero,[20] and (3) *polar*, meaning that the quantity has both positive and negative values. These properties are formally specified below (let ω be a function from an individual to its quantity value, ε be a real number, and each v_i be a quantity value):

SUMMATIVE:	$\omega(a_1, a_2) = \omega(a_1) + \omega(a_2) - \omega(a \cap b)$
CONVERGEABLE:	$\forall \varepsilon > 0, \exists v_n (v_n < \varepsilon)$
POLAR:	$\forall v_1 \exists v_2 (v_1 + v_2 = 0)$

[20] Convergeability is tricky. In most (maybe all) cases, only a finite set of values can be instantiated by the kinds of objects that actually exist. But that's compatible with thinking that there are possible values of the quantity that aren't instantiated in the actual world. For example, even if there is a minimal weight instantiated by actual objects, perhaps there are still smaller weight values that aren't instantiated by any actual objects. This is how I'll think about the quantities under consideration.

None of these conditions is guaranteed to be satisfied. As examples, it's arguable that (*a*) volume is summative and convergeable but not polar, (*b*) wealth is summative and polar but not convergeable,[21] (*c*) height-above-sea-level is polar and convergeable but not summative, and (*d*) number-of-children is neither summative, convergeable, nor polar.[22]

Philosophical work on quantities has focused mostly on what quantities have in common (and what distinguishes them from non-quantitative properties). But the differences above illustrate the diversity that exists within the domain of quantities. An interesting project would be to identify the most interesting features that differentiate classes of quantities in order to generate a general taxonomy.

§5 Supertasks

Many philosophical puzzles concerning infinitary scenarios involve *supertasks*—scenarios where an infinite number of steps are completed within a finite amount of time. The goal of this last section is to illustrate how the expansionist analysis sheds light on puzzles about supertasks. To start, let's return to a principle that was mentioned earlier:

> **CONTINUITY**
> If a quantity converges to a limit x over time interval $[t_0, t_1)$, then the quantity attains value x at t_1.

Hoek [2021] warns that CONTINUITY isn't a reliable guide to the outcomes of supertasks. He says: "We cannot uncritically apply the Continuity

[21] This assumes that wealth is measured in a currency with a minimal value: for example, the minimal unit for US dollars is 1 cent. If we instead consider an infinitely divisible currency, such as bitcoin, then wealth may be convergeable.

[22] Why isn't number-of-children summative? Suppose Alice's children are Ivan, Jade, and Kane, while Bob's children are Kane and Lyra. Then Alice has 3 children and Bob has 2 children, but Alice and Bob altogether have 4 children.

Principle…[T]he answer is different in each case…[E]ach supertask raises its own, subject-specific set of questions" (*p*.4). Although I think Hoek is right that CONTINUITY doesn't always yield the right results, I also think that we can systematically diagnose when the principle holds and when it doesn't. The answer depends on whether time is a locative category for the quantity under consideration.

Any supertask will take place over some interval of time $[t_0, t_1)$, such that the supertask begins at t_0 and is complete at t_1. To apply the expansionist analysis to a supertask, we need to consider increasingly large temporal intervals (t_j, t_k) such that for all k, t_k is before t_1. In other words, we appeal to ever-expanding intervals of time that approach (but don't reach) the end-time of the supertask. However, before we get to that point, we must first ask whether the expansionist analysis is even appropriate for the supertask at hand. To do that, we need to figure out whether the supertask involves summation over some quantity, and if so, ask whether time is a locative category for that quantity. In what follows, I'll discuss three supertasks and show how each warrants relevantly different analyses.[23]

Infinite Flea

Items: A flea jumping around on a line.

[23] Another philosophical puzzle is the Pasadena game, introduced by Nover & Hajek [2004]. Imagine you are presented with the following game: a coin is flipped n times, where n = the first flip where the coin comes up heads. If n is odd, then you pay $2^n/2$ dollars. If n is even, then you receive $2^n/2$ dollars. How much should you be willing to pay to play this game? If we calculate the expected value for each value of n, then we encounter a familiar series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = ln(2)$. But Nover & Hajek argue that any order by which we sum these expected values is arbitrary, leading them to conclude that the value of the Pasadena game is indeterminate. The Pasadena game raises some tricky issues that warrant more discussion than the other puzzles discussed in this section. For limits of space, I won't address it in this paper.

Setup: The flea starts at 0. At 1pm, it jumps 1cm to the right. At 1:30pm, it jumps $\frac{1}{2}$ cm to the left. At 1:45pm, $\frac{1}{3}$ cm to the right. And so forth.

Just as with *Infinite Scale*, the movements of the flea can be modeled by the alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = ln(2)$. Linnebo [2020] suggests that *Infinite Flea* is merely another case where the result of an infinitary scenario depends on the order in which the individual items are evaluated—instead of iron balls and balloons, the relevant items are now left-jumps and right-jumps. This means that the order-relative analysis endorses CONTINUITY (at least for *Infinite Flea*). Since we've seen that appeals to temporal ordering relations can yield implausible results in some infinitary scenarios, it's reasonable to be suspicious of this way of reasoning about *Infinite Flea*. But this suspicion can be assuaged by observing an important asymmetry between *Infinite Scale* and *Infinite Flea*.

Think of the quantity in *Infinite Flea* as distance traveled, where jumps to the right are positive distances and jumps to the left are negative distances. To elicit the asymmetry between *Infinite Scale* and *Infinite Flea*, consider the contrast between the following questions:

(3*a*) How much distance was traveled in this interval of time?
(3*b*) How much weight was there in this interval of time?

Although time isn't a locative category for weight, it *is* for distance traveled. In other words, it makes sense to sum distance traveled (but not weight) over intervals of time. Given this, we ought to expect CONTINUITY to yield the right results for *Infinite Flea*. This is because if we take time to be the relevant locative category, then an invocation of the expansionist analysis is effectively an invocation of CONTINUITY. Both CONTINUITY and the expansionist analysis appeal to convergence to a limit: it's just that CONTINUITY requires that the limit is defined over intervals of time, whereas the expansionist analysis leaves open which locative category is relevant. For *Infinite*

Flea, the question becomes whether the amount of distance traveled via the finite subsets of jumps occurring over ever-expanding intervals of time (within $[t_0, t_1)$) always approaches some limit x. The answer is 'yes': that limit is $ln(2)$.[24]

Consequently, the order-relative analysis happens to be right about *Infinite Flea:* the total distance traveled by the flea depends on the order in which the flea makes its jumps. But the expansionist analysis gives a deeper explanation for why the order-relative analysis works in this case: namely, because time is the relevant locative category in this scenario. However, that condition may not hold for other supertasks involving other quantities. In fact, we will now turn to one such case.

The Ross-Littlewood Paradox

Items: A jar that can hold infinitely many balls, and a countably infinite pile of balls, numbered 1, 2, 3, and so forth.

Setup: At t_0, we place balls 1–10 into the jar. Then we remove ball 1. Then we add balls 11–20 into the jar. Then we remove ball 2. We repeat indefinitely. By time t_0, every ball from the original infinite pile has been placed in the jar.

How many balls are in the jar once the supertask is complete? If we appeal to CONTINUITY, then it seems that we should conclude that the answer is ∞. For every time before t_1, the number of balls in the jar grows increasingly

[24] More precisely, we get $ln(2)$ so long as we choose a center p within the interval [0, .5], since choosing a center within that subinterval ensures that every jump within [0, 1) will eventually be covered by the expanding intervals of time centered on p. If p is instead within the interval (.5, 1), then there will be a subinterval of [0, 1) that is never reached by the uniform expansions over intervals of time. But then we can simply take the result of the expansionist analysis and add that to the finite sum associated with the additional subinterval, in which case we would again get $ln(2)$ as the total sum.

large. If we mathematically represent the number of balls that are added to or removed from the jar at each step, we get the series $10 - 1 + 10 - 1 + \cdots$, which clearly diverges to ∞. However, Littlewood [1953] and Ross [1976] both argue that the answer is 0. The reason is that every ball is eventually removed from the jar. That is, for every ball in the jar, there will be some time before t_1 when that ball is removed. If one were to say that the number of balls at t_1 is ∞, we could ask *which* balls are in the jar at t_1. But any ball we pick would eventually be out of the jar at some time, so it seems that there's no ball such that it remains in the jar at t_1.

We can get insight into the Ross-Littlewood paradox by appealing to the expansionist analysis. The quantity under consideration is number-of-balls. Should we expect the value of this quantity at t_1 to be the limit value of the quantity for the times before t_1? Well, we can ask whether time is a locative category for number of balls:

(4*a*) How many balls are in this interval of time?
(4*b*) How many balls are in this region of space?

The asymmetry is evidence that time *isn't* a locative category for number-of-balls. Given this, we ought to refrain from appealing to CONTINUITY when assessing the Ross-Littlewood Paradox. Now, that's merely a negative result: it doesn't yet settle how many balls are in the jar at t_1. But it undermines the main motivation for thinking that the answer is ∞. On the other hand, the argument that every ball is eventually removed from the jar remains untouched. Given this, I think the most reasonable answer to the Ross-Littlewood Paradox is 0.

Let's turn now to the last supertask:

Thomson's Lamp

Items: A lamp.

Setup: The lamp is turned on at 1:00pm, then off at 1:30, on at 1:45, and so on.

What's the state of the lamp at 2pm? The available answers are 'on', 'off', or that the scenario is under-described. I favor 'under-described': it seems to me that the specification of the scenario simply leaves open whether the lamp is on or off at 2pm. But set that aside: I want to instead make a more general point about how we reason about scenarios like Thomson's Lamp.[25]

It's often thought that Thomson's Lamp can be modeled by the infinite series $1 - 1 + 1 - 1 + \cdots$, often known as 'Grandi's Series'. But there's an important asymmetry between Thomson's Lamp and the other scenarios we've considered: the variable property in Thomson's Lamp— the states 'on' and 'off'—aren't values along a quantity. Consider how Thomson's Lamp could just as well be defined as a scenario where the lamp flips back and forth between red light and green light, or as a screen that alternates between displaying 'A' and 'B'. Given this, it strikes me as inappropriate to model Thomson's Lamp using Grandi's Series. If we wish to represent a scenario with an infinite series, then we should first ensure that the scenario involves modulations of some quantity. Otherwise, there won't be a meaningful interpretation of the addition and subtraction operations, and we risk conflating features of the mathematical representation with features of the scenario being represented.

Granted, there *is* an obvious quantity that we could focus on in this scenario: namely, luminosity. We might interpret the luminosity level as 1 when the lamp is on and 0 when the lamp is off. Then the question becomes whether time is a locative category for luminosity. Does it make sense to ask how much luminosity there is over an interval of time? I myself feel unsure in this particular case. But let's conditionalize. If time *isn't* a locative

[25] This scenario originates from Thomson [1954], which is also where the term 'supertask' was first introduced. The idea that the scenario is underspecified is often associated with Benacerraf [1962].

category for luminosity, then that means that even when we reinterpret Thomson's Lamp as involving modulations in luminosity levels, it's *still* inappropriate to use Grandi's Series to model Thomson's Lamp. On the other hand, if it does make sense to sum luminosity over time, then it's clear that Grandi's Series will be the appropriate mathematical representation for this case. And since Grandi's Series diverges, we might then conclude that the luminosity level of Thomson's Lamp at 2pm is undefined.

Now, it's actually possible to resist that conclusion. Throughout this paper, I've taken for granted the standard definition of the sum of an infinite series, where the sum is the limit of the sequence of partial sums of that series. But there are more powerful mathematical methods that assign finite numbers even to series that diverge under standard summation.[26] For Grandi's Series, nearly every one of the more sophisticated summation methods assign a sum of ½. And that brings us to a whole new philosophical question about infinite sums: which mathematical method for summation best captures the metaphysics of physical quantities?

I think that's a fascinating question. But it's a question that will have to be reserved for another time.

Conclusion

The puzzle of conditional convergence may initially seem like a remote curiosity. But the solution to the puzzle is surprisingly consequential. Though I've focused on weight, the puzzle generalizes to any quantity that is summative, convergeable, and polar. And the solution I favor—the expansionist analysis—has implications well beyond our initial puzzle. I've explained how it provides a diagnosis for when CONTINUITY yields the right verdicts in a supertask and when it doesn't. And although I haven't had space to discuss other philosophical problems, I suspect there will be further applications to other infinitary puzzles in ethics and decision theory.

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[26] See Hardy [1992].

To properly solve the puzzle of conditional convergence, we needed to explore the general connections between quantities and locations. Only then were we in position to fully appreciate the metaphysical significance of Riemann's Rearrangement Theorem. On the picture I've developed, quantities are indexed to both categories of individuals (namely, the individuals that can bear values along that quantity) and categories of locations (namely, the locations at which that quantity can be instantiated). And summation over locations and summation over individuals interact in systematic ways, as illustrated by QUANTIFICATION EQUALITY, SUMMATION OVER REGIONS, and SUMMATION OVER INDIVIDUALS.

To my knowledge, there has been little prior philosophical investigation into the relationships between quantities and locations. The philosophical literature on quantities tends to focus on the structural features that distinguish quantities from other kinds of properties and on the ontology of quantities. The philosophical literature on locations tends to focus on the formal principles connecting locations to mereology and on the debates between substantivalists and relationalists. Hence, the puzzle of conditional convergence—and the expansionist analysis, in particular—points towards a line of metaphysical inquiry that is ripe for exploration.

There are many interesting questions that remain open. I've focused on weight, but we might also ask what the relevant locative categories are for other quantities. I've focused on metric spaces, but we might also ask whether the expansionist analysis can be extended to more general spaces, such as topological spaces. I've focused on uniform expansions, but we might also ask whether there are quantities for which non-uniform expansions yield the right results. I've focused on infinitary regions, but we might also generalize the expansionist analysis to cases where infinitely many items lie within a finite region. And I've assumed that summation is simply a matter of the limits of partial sums, but we might ask whether a more powerful mathematical method for summation will be more metaphysically apt. These all strike me as promising lines for future research.

APPENDIX: Riemann's Rearrangement Theorem

A *sequence* is an ordered list of numbers. A *series* is the sum of all the terms in a sequence. A series *converges* iff there exists a real number l such that the sequence of partial sums of the series converges to l. This is equivalent to saying that a series *converges* iff for any $\varepsilon > 0$, there exists an integer m such that for all $n \geq m$, the difference between l and the partial sum of the first n terms of the series is less than ε.

Some series are *absolutely convergent*, meaning that the order of the terms in the series doesn't make any difference to the sum of the series. Other series are *conditionally convergent*, meaning that the sum of the series depends on the order of its terms. More precisely, a series is conditionally convergent iff that series converges yet the series consisting of the absolute values of its terms diverges. That is:

Definition: $\sum a_n$ *conditionally converges* $=_{\text{def}}$ $(\exists l : \sum a_n = l)$ and $(\neg\exists l : \sum |a_n| = l)$.

As an example, consider again the alternating harmonic series:

The Alternating Harmonic Series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln(2)$$

This series converges to $ln(2)$. But it's conditionally convergent: if we take the absolute values of its terms, then the resultant series $|\,1\,| + |-\frac{1}{2}\,| + |\,\frac{1}{3}\,| + |-\frac{1}{4}\,| + \cdots$ diverges to ∞. Notably, whenever an infinite series is conditionally convergent, the series containing all and only its positive terms diverges to ∞, and the series containing all and only its negative terms diverges to $-\infty$. That fact follows from the definition of 'conditional convergence', and will be important in what follows. Next, let's turn to Riemann's Rearrangement Theorem:

Riemann's Rearrangement Theorem:
If an infinite series is conditionally convergent, then its terms can be rearranged so that the new series converges to an arbitrary number, or diverges.

In what follows, I'll explain how to make the rearranged series sum to an arbitrary positive number or diverge to ∞. It will be straightforward to generalize to the cases involving arbitrary negative numbers and −∞.

To start, let's extract from the alternating harmonic series the series consisting of all and only its positive terms and the series consisting of all and only its negative terms. Note that for both the positive series and the negative series, the terms grow arbitrarily close to zero as the series progresses:

The Positive Series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$$

The Negative Series

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots = -\infty$$

Suppose we wish to rearrange the alternating harmonic series so that it converges to an arbitrary positive number l. We start by taking terms from the positive terms until the sum of those terms exceeds l. Then we take terms from the negative series until the sum goes below l. Then we continue the procedure, moving to the positive series whenever the sum goes below l, then the negative series whenever the sum exceeds l, and so on. Since the original alternating harmonic series converged, following this procedure guarantees that the rearranged series will converge. The result is a rearranged series that contains all and only the terms of the original series, yet which converges to l.

Suppose we wish to rearrange the alternating harmonic series so that it diverges to ∞. We start by taking the first term in the negative series. Then, we add terms in the positive series until the resultant sum is greater than 1. Next, we add the second term in the negative series. Then, we add terms in the positive series until the resultant sum is greater than 2. Since the positive series diverges to ∞, it's guaranteed that our remaining set of positive terms will suffice to exceed any finite positive integer, no matter how far along we are in the procedure. We repeat this procedure indefinitely. The result is a rearranged series that contains all and only the terms in the original alternating harmonic series, yet which diverges to ∞.

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